A Hybrid Iterative approach for solving the linear systems from interior point methods

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Summary

- Interior Point Methods
- Linear System Solution
- Iterative Methods
- Preconditioners
- Implementation Issues
- Computational Results
- Concluding Remarks

Linear Programming Problem

- Primal Problem

$$\begin{aligned} \text{min} \quad & c^t x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Dual Problem

$$\begin{aligned} \text{max} \quad & b^t y \\ \text{subject to} \quad & A^t y + z = c \\ & z \geq 0 \end{aligned}$$

- Optimality Conditions

$$\begin{aligned} Ax - b &= 0 \\ A^t y + z - c &= 0 \\ XZe &= 0 \\ (x, z) &\geq 0 \end{aligned}$$

Interior Point Methods

- Affine Directions

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^t & I \\ Z^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta\tilde{x}^k \\ \Delta\tilde{y}^k \\ \Delta\tilde{z}^k \end{bmatrix} = \begin{bmatrix} r_p^k \\ r_d^k \\ r_a^k \end{bmatrix}$$

$$\begin{cases} r_p^k & = & b - Ax^k \\ r_d^k & = & c - A^t y^k - z^k \\ r_a^k & = & -X^k Z^k e \end{cases}$$

Interior Point Methods – Predictor Corrector Version

- Search Directions

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^t & I \\ Z^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta y^k \\ \Delta z^k \end{bmatrix} = \begin{bmatrix} r_p^k \\ r_d^k \\ r_c^k \end{bmatrix}$$

$$r_c^k = \mu^k e - X^k Z^k e - \Delta\tilde{X}^k \Delta\tilde{Z}^k e$$

Linear System Solution

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^t & I \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} r_p \\ r_d \\ r_a \end{bmatrix}$$

- Augmented System

$$\begin{bmatrix} -D^{-1} & A^t \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} r_d - X^{-1} r_a \\ r_p \end{bmatrix}$$

where $D = Z^{-1} X$.

- Normal Equations System

$$(ADA^t)\Delta y = AD(r_d - X^{-1} r_a) + r_p$$

Splitting Preconditioner

Preconditioned Matrix $K = -I_n + D^{-\frac{1}{2}} A^t G^t + G A D^{-\frac{1}{2}}$
where $G = H^t D_B^{\frac{1}{2}} B^{-1}$ and $H = [I_m \; 0] P$
P is a permutation matrix and $A = [B \; N] P$.

- Avoid Computing the Normal Equations System
- LU Factorization of B
- Dimension n
  - m Positive Eigenvalues
  - $n - m$ Negative Eigenvalues
- MINRES
- $G^t D^{-\frac{1}{2}} A^t = A D^{-\frac{1}{2}} G = I_m$

$$\begin{aligned} K = -I_n \quad &+ \quad U^t V^t \quad + \quad VU \\ \text{where } UV \quad &= \quad V^t U^t \quad = \quad I_m \end{aligned}$$

Normal Equations Version

- $A = [B\ N]P$;

$$ADA^t = BD_BB^t + ND_NN^t$$

$$D_B^{-\frac{1}{2}}B^{-1}ADA^tB^{-t}D_B^{-\frac{1}{2}} = I_m + D_B^{-\frac{1}{2}}B^{-1}ND_NN^tB^{-t}D_B^{-\frac{1}{2}}$$

- No Need to Compute ADA^t
- LU Factorization of B
- Dimension m
- Positive Definite
- Conjugate Gradient Method
- MINRES
  - Careful Implementation
  - Implicit Restart

Implicit Restart

$$\begin{aligned}
\theta^+ V_j Q e_1 &= \theta V_j e_1 - \tau A V_j e_1 \\
\theta^+ V_j Q e_1 &= \theta V_j e_1 - \tau (V_j H_j + f_j e_j^t) e_1 \\
\theta^+ Q e_1 &= \theta e_1 - \tau (QR + \mu I) e_1 \\
\theta^+ Q e_1 &= \theta e_1 - \tau (\rho_{11} Q + \mu I) e_1
\end{aligned}$$

hence $\tau = \frac{\theta}{\mu}$ and

$$\theta^+ = -\theta \frac{\rho_{11}}{\mu} \text{ and } x_0^+ = \frac{1}{\mu} r$$

- Compute the norm of the new residual without computing the residual itself
- p shifts at once obtaining a k-step
- Ritz values or harmonic Ritz values as shifts

Implementation Issues

- Indefinite System
- Normal Equations System
- Finding a Permutation P
- Rules for Reordering the Columns of A
  - Diagonal value of D
  - $||DAe_j||_1$
  - $||DAe_j||_2$
- Compute LU Factorization to Determine the Columns of B
- Careful Implementation
- No Need to Compute ADA^t
- Works Fine near a Solution
- Designed for Last IP Iterations
- It is not Efficient far from a Solution
- First IP Iterations

Computational Results

- **Computer:** Intel Core i7, 2.93GHz, 16GB RAM, HD 1TB.
- **Operational System:** Linux 64Bits.
- **Language:** C.
- Cplex to read the problems.
- Splitting Preconditioner to accelerate the convergence of iterative methods (MINRES, CG and hybrid approach).
- **Test Problems:** 66 (NETLIB, QAPLIB, KENNINGTON, STOCHLP).

Computational Results

Augmented System

- PCG: does not achieve convergence.
- PMINRES: robust, achieve good results.
  - **Without Refactorization:** 24 problems solved.
  - **With Refactorization:** 25 problems solved.

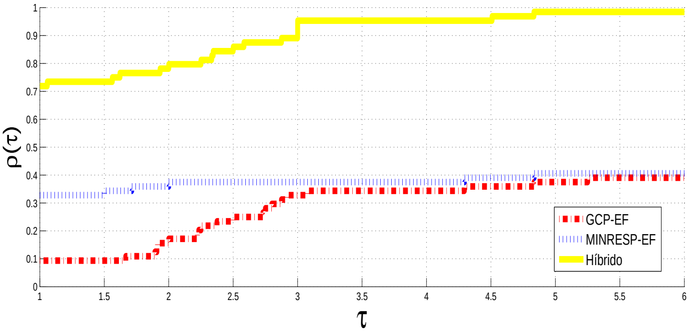
Computational Results

PROBLEM	DIMENSION		PCG-EF		PMINRES-EF		HYBRID-EF	
	ROW	COLUMN	T(s)	It.	T(s)	It.	T(s)	It.
afiro	23	47	0,00	9	0,00	9	0,00	9
agg	355	443	1,20	36	*	*	0,39	23
fit1d	24	1049	*	*	0,06	24	0,29	25
fit2d	25	10524	*	*	2,48	88	11,18	57
kb2	39	64	0,01	11	*	*	0	11
ken-07	1272	2448	0,60	16	0,27	16	0,54	16
nug05	210	225	0,70	8	0,03	8	0,07	8
nug07	602	931	5,19	12	4,66	14	2,72	12
pds-02	2499	7229	4,69	29	2,10	29	4,06	29
qap8	912	1632	18,73	10	*	*	9,79	10
recipe	64	150	0,02	11	0,00	11	0,03	11
sc50a	38	66	0,00	10	0,01	10	0,00	10
sc50b	37	65	0,00	9	0,00	10	0,00	9
sc105	83	141	0,01	10	0,01	10	0,01	10
sc205	164	276	0,21	18	*	*	0,04	11
scagr7	72	128	0,01	10	0,01	16	0,01	16

Computational Results

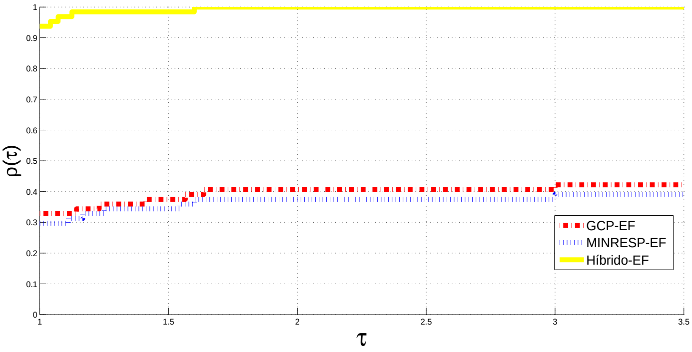
PROBLEM	DIMENSION		PCG-EF		PMINRES-EF		HYBRID-EF	
	ROW	COLUMN	T(s)	It.	T(s)	It.	T(s)	It.
scsd1	77	760	*	*	0,02	11	0,06	11
scsd6	147	1350	0,14	11	0,05	11	0,15	11
scsd8	397	2750	0,52	11	*	*	0,55	11
sctap1	284	644	0,23	19	0,08	19	0,2	19
sctap2	1033	2443	2,25	24	0,29	17	0,68	17
shell	392	1383	0,53	23	*	*	0,28	23
ship04l	317	1955	0,18	15	0,08	15	0,23	15
ship04s	241	1331	*	*	0,21	20	0,14	16
ship08l	520	3221	*	*	0,37	16	0,6	18
ship08s	326	1704	0,22	16	0,08	16	0,24	16
ship12l	687	4325	0,71	18	0,26	18	0,78	18
ship12s	417	2097	0,30	17	0,12	17	0,31	17
standgub	287	1165	0,57	22	0,24	22	0,54	22
standmps	395	1165	1,40	32	0,48	28	0,75	30
stocfor1	89	137	0,05	16	*	*	0,03	13

Computational Results



Running Time - Normal Equations without LU refactorization.

Computational Results



Number of iterations - Normal Equations without LU refactorization.

Computational Results

Normal Equations System without LU Refactorization:

- Problems solved: HA 97%, PMINRES 36% and PCG 39%.
- Running Time: HA faster than PCG and PMINRES.
- Iteration: HA needs less iterations to converge.

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ganges	952	1348	1,52	20	*	*	1,39	20
kb2	39	64	0,02	11	*	*	0	11
ken-07	1272	2448	0,61	16	0,75	16	0,54	16
nug05	210	225	0,08	8	0,09	8	0,07	8
nug06	372	486	0,42	9	0,52	9	0,43	9
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pds-02	2499	7229	4,68	29	*	*	4,06	29
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scagr7	72	128	0,01	10	0,01	16	0,01	16
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sctap3	1408	3268	66,2	91	*	*	11,25	20
share2b	93	159	0,11	19	0,12	19	0,04	12
shell	392	1383	0,57	23	0,94	24	0,28	23
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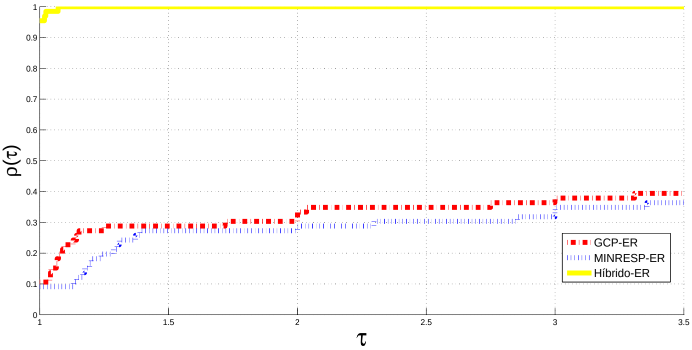
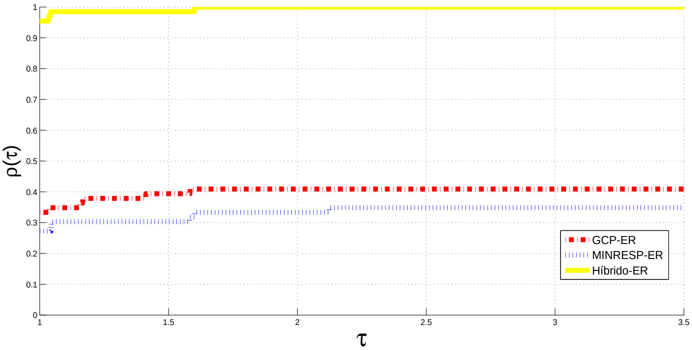
ship12s	417	2097	0,33	17	0,37	17	0,31	17
standgub	287	1165	0,59	22	1,57	23	0,55	22
standmps	395	1165	0,79	30	1,7	29	0,74	30
stocfor1	89	137	0,04	15	0,36	54	0,02	13
truss	1000	8806	12,45	18	*	*	11,44	18

Computational Results



Running Time - Normal Equations with LU refactorization.

Computational Results



Number of iterations - Normal Equations with LU refactorization.

Computational Results

Normal Equations System with LU Refactorization:

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Concluding Remarks

- MINRES;
- Indefinite System;
- Hybrid Preconditioner Approach;
- Preconditioners Transition;
- Methods Transition;
- Robustness and Speed